

# Near horizon symmetry and entropy of black holes in the presence of a conformally coupled scalar

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## Abstract

We analyze the near horizon conformal symmetry for black hole solutions in gravity with a conformally coupled scalar field using the method proposed by Majhi and Padmanabhan recently. It is shown that the entropy of the black holes of the form  $ds^2 = -f(r)dt^2 + dr^2/f(r) + \dots$  agrees with Wald entropy. This result is different from previous result obtained by M. Natsuume, T. Okamura and M. Sato using the canonical Hamiltonian formalism, which claims a discrepancy from Wald entropy.

## 1 Introduction

Holographic duality is one of the most interesting subject in high energy physics since 't Hooft and Susskind[1, 2] proposed the so-called holographic principle. The most well-known realization of holographic duality is through AdS/CFT. Though the idea of AdS/CFT becomes popular only after Maldacena's celebrated work [3], the first known example of CFT dual of AdS gravity was actually constructed before Maldacena's work. Early in 1986, Brown and Henneaux found the conformal symmetry on the asymptotic boundary of AdS spacetime using the canonical Hamiltonian formalism [4]. Using this result, Strominger gave an explanation of the microscopic origin of  $(2+1)$ -dimensional black hole entropy [5] with the aid of Cardy's formula [6], and the result agrees with Bekenstein-Hawking entropy. Instead of working at the asymptotic infinity, Carlip extended such studies to the near horizon region of black hole solutions [7, 8], and also obtained Bekenstein-Hawking entropy through Cardy's formula. Subsequent works are too numerous to be listed here, though it is worth mentioning the recent development in the studies of Kerr/CFT correspondence, which generalizes the result to axisymmetric spacetimes [9, 10, 11]. Most of the works mentioned above take the view point that

the near horizon conformal symmetry is the consequence of the asymptotic symmetry of the total action of the system, i.e. bulk plus boundary contribution.

Recently, Majhi and Padmanabhan [12] proposed a new method (MP approach for short) for constructing the near horizon conformal symmetry. The Noether current they constructed is related to diffeomorphism invariance of the *boundary action* alone, as apposed to the whole theory. This different construction is reasonable, because it is natural to expect that the entropy of the black hole is originated only from the degrees of freedom around or on the relevant null surface. The Killing vectors that leave the boundary action invariance under diffeomorphism is chosen to keep the form of the metric near the horizon in order to ensure that the horizon is not destroyed by the diffeomorphism. A direct calculation shows that the algebra of the conserved charges is Virasoro algebra, whose central charge can be inserted into Cardy's formula to get the black hole entropy. For pure gravity without matter source, the resulting entropy matches the Bekenstein-Hawking entropy.

A natural question one wants to ask is if the quantum nature of a black hole is also captured by a dual conformal field theory when matter source come in, the simplest example being the black hole solutions in gravitational theories coupled with a scalar field. For minimally coupled scalar, existing studies show that the central charge is the same as the pure gravity [13]. As to the conformally coupled case, through the canonical Hamiltonian formalism proposed by Brown and Henneaux, it was shown [14] that the *central charge* thus obtained is the same as in the pure gravity case, consequently the entropy *disagrees* with the Wald entropy [15]. This is a quite weird situation, because it is generally believed that the Wald entropy is universal for gravitational theories with higher curvature and/or matter source contribution. In this paper, we are aimed to reconsider the near horizon symmetry and entropy of black holes in the presence of a conformally coupled scalar field using the MP approach. It turns out that the near horizon symmetry still closes a Virasoro algebra, however, the central charge receives a contribution from the scalar field and is different from that one would expect in pure gravitational theories or in the case with a minimally coupled scalar field. However, the entropy obtained using Cardy's formula does agree with the Wald entropy.

This paper is organized as follows: In section 2, we briefly review the MP approach for constructing the Noether current arising from invariance of the boundary action, taking pure Einstein gravity as an example. In section 3, we study the conformal symmetry in the near horizon region of a special class of black hole solution of the form (12) in the presence of a conformally coupled scalar field and calculate the black hole entropy using Cardy's formula. It turns out that the Virasoro central charge is different from the minimally coupled case. However, the entropy matches the Wald entropy. The last section of the paper is devoted to the summarization of the results.

## 2 Boundary action and MP approach

The MP approach relies purely on the *boundary action* of the gravitational theory under consideration. As is well known, the variational process of a gravitational theory requires that there is a boundary counter term which cancels out the total divergence term arising from the variation of the bulk Lagrangian. This boundary counter term is what is referred to as the boundary action in this paper.

According to Stokes theorem, any boundary action can be rewritten as a bulk integral. Omitting the integration symbol, we can identify the integrand as

$$\sqrt{g}L = \sqrt{g}\nabla_a A^a, \quad (1)$$

where  $A^a$  is a vector field defined on the bulk spacetime. Under a diffeomorphism  $x^a \rightarrow x^a + \xi^a(x)$ , the variation of the left hand side of (1) is given by

$$\delta_\xi(\sqrt{g}L) \equiv \mathcal{L}_\xi(\sqrt{g}L) = \sqrt{g}\nabla_a(L\xi^a), \quad (2)$$

where  $\mathcal{L}_\xi$  denotes the Lie derivative along the vector field  $\xi^a(x)$  and we have  $\mathcal{L}_\xi(\sqrt{g}) = \sqrt{g}\nabla_a \xi^a$  and  $\mathcal{L}_\xi(L) = \xi^a \nabla_a L$ . Similarly, the right hand side changes by

$$\begin{aligned} \delta_\xi(\sqrt{g}\nabla_a A^a) &= \mathcal{L}_\xi[\partial_a(\sqrt{g}A^a)] \\ &= \partial_a[A^a \mathcal{L}_\xi \sqrt{g} + \sqrt{g} \mathcal{L}_\xi A^a] \\ &= \sqrt{g}\nabla_a [\nabla_b(A^a \xi^b) - A^b \nabla_b \xi^a], \end{aligned} \quad (3)$$

where  $\nabla_a A^a = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}A^a)$  is used.

Equating eq.(2) and eq.(3), we get the conservation of the following Noether current,

$$J^a[\xi] = L\xi^a - \nabla_b(A^a \xi^b) + A^b \nabla_b \xi^a.$$

Inserting (1) into this formula, we get

$$J^a[\xi] = \nabla_b J^{ab}[\xi], \quad (4)$$

where

$$J^{ab}[\xi] = \nabla_b [\xi^a A^b - \xi^b A^a] \quad (5)$$

is known as the *Noether potential*. As we have seen, the Noether current  $J^a[\xi]$  arises purely from variations of the the boundary action rather than from the bulk one.

For pure Einstein gravity without matter source, the boundary action is given by the well-known York-Gibbons-Hawking term

$$\begin{aligned} I_B &= \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^d x \sqrt{\sigma} K \\ &= \frac{1}{8\pi G} \int_{\mathcal{M}} d^{d+1} x \sqrt{g} \nabla_a (K N^a), \end{aligned} \quad (6)$$

where  $N^a$  is the unit vector field normal to the boundary hypersurface  $\partial\mathcal{M}$  and  $K = -\nabla_a N^a$  is the trace of the extrinsic curvature. The Noether current is now given by

$$J^a[\xi] = \nabla_a J^{ab} = \frac{1}{8\pi G} \nabla_b (K \xi^a N^b - K \xi^b N^a). \quad (7)$$

The corresponding Noether charge is defined as

$$Q[\xi] = \frac{1}{2} \int_{\partial\Sigma} \sqrt{h} d\Sigma_{ab} J^{ab}, \quad (8)$$

where  $d\Sigma_{ab} = -d^{(d-1)}x (N_a M_b - N_b M_a)$  is the area element on the  $(d-1)$ -dimensional hypersurface  $\partial\Sigma$ .  $N^a$  and  $M^a$  are respectively spacelike and timelike unit normal vectors.

Finally, the algebra of the conserved charges is defined as:

$$\begin{aligned} [Q_1, Q_2] &\equiv \frac{1}{2} (\delta_{\xi_1} Q[\xi_2] - \delta_{\xi_2} Q[\xi_1]) \\ &= \frac{1}{2} \int_{\partial\Sigma} \sqrt{h} d\Sigma_{ab} [\xi_2^a J^b[\xi_1] - \xi_1^a J^b[\xi_2]], \end{aligned} \quad (9)$$

where  $\delta_{\xi_1} Q[\xi_2] = \int_{\Sigma} d\Sigma_a \mathcal{L}_{\xi_1} (\sqrt{g} J^a[\xi_2])$ . As was shown in [12], this algebra leads to the Virasoro algebra in the near horizon limit. In the next section, we shall show that similar constructions also works for a special class of black hole solutions in gravitational theories with a non-minimally coupled scalar field.

### 3 Near horizon symmetry in the presence of a conformally coupled scalar field

In this section, we consider the near horizon symmetries of a static black hole with a conformally coupled scalar hair. The desired Virasoro algebra can be derived using the MP approach, from which the entropy is given with Cardy's formula. The result agrees with Wald entropy as will be seen.

The bulk action of gravity with a conformally coupled scalar field reads

$$I = \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left[ \frac{1}{16\pi G} \left( R - \frac{d(d-1)}{\ell^2} \right) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{d-1}{8d} R \phi^2 - U(\phi) \right], \quad (10)$$

where  $\ell$  is the AdS radius and  $U(\phi)$  is the scalar potential. The scalar field  $\phi$  couples to gravity not only through the kinetic term but also through the  $R\phi^2$  term. With the special value  $\frac{d-1}{8d}$  of the curvature coupling constant, the kinetic term for the scalar field and the curvature coupling term put together are invariant under conformal rescaling

of the metric field, that is why this special value of curvature coupling is referred to as conformal coupling. The boundary term associated with the above action is [16]

$$I_B = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^d x \sqrt{h} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2 \right) K. \quad (11)$$

In deriving the above boundary term, we take the usual boundary conditions that keep the metric and scalar field fixed but not their normal derivatives, i.e.  $\delta g_{\mu\nu} = \delta\phi = 0$  on  $\partial\mathcal{M}$ , while  $n^\rho \partial_\rho \delta g_{\mu\nu}$  and  $n^\rho \partial_\rho \delta\phi$  do not vanish. From the bulk action (10), we see that, only the kinetic term contributes to the boundary term after taking variation, which is proportional to  $\delta\phi n^\mu \nabla_\mu \phi$ , thus we do not need a boundary term to cancel it. The variation with respect to the scalar field of the boundary term (11) also vanishes. As to the variation of the metric, everything goes like the case of pure gravity with only an additional factor  $(1 - \frac{d-1}{4d} 8\pi G \phi^2)$ . As a confirmation, one can take a conformal transformation that transforms the non-minimally coupled case to the minimal case, then the boundary term becomes the standard York-Gibbons-Hawking term, as can be seen in [16, 17, 18].

Now we consider a static black hole metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (12)$$

where  $d\Omega^2$  is the line element on a unit  $(d-1)$ -sphere. Though not being the most general form of static, spherically symmetric metrics, the above form of metric ansatz is very frequently adopted in pure gravity (e.g. in Schwarzschild metric), gravity with minimally coupled matter sources (see, e.g. [19]) as well as in gravity with nonminimally coupled sources [20, 21, 22]. Note that the black hole metric given in [20] is just the one used in [14], which led to an entropy which is different from Wald entropy using the canonical Hamiltonian formalism.

Suppose that there is an event horizon at  $r = r_h$ , i.e.  $f(r_h) = 0$ . In order to consider the near-horizon region, it is convenient for us to introduce a new coordinate  $\rho$  via  $r = \rho + r_h$ , in terms of which the metric becomes

$$ds^2 = -f(\rho + r_h)dt^2 + \frac{d\rho^2}{f(\rho + r_h)} + (\rho + r_h)^2 d\Omega^2 \quad (13)$$

In the near horizon approximation, i.e.  $\rho \rightarrow 0$ ,  $f(\rho + r_h)$  can be expanded as  $f(\rho + r_h) = 2\kappa\rho + \frac{1}{2}f''(r_h)\rho^2 + \dots$ , with  $\kappa = \frac{f'(r_h)}{2}$  being the surface gravity. In the leading order approximation, the well known Rindler metric can be achieved  $ds^2 = -2\kappa\rho dt^2 + \frac{1}{2\kappa\rho}d\rho^2 + \dots$ .

In order to obtain a Virasoro algebra, the vector field  $\xi^\mu$  should be taken properly. It is convenient for us to transform to the Bondi-like metric to solve the Killing equations and then transform back. Using the coordinate transformation

$$du = dt - \frac{d\rho}{f(\rho + r_h)}, \quad (14)$$

the metric (13) can be rewritten as

$$ds^2 = -f(\rho + r_h)du^2 - 2dud\rho + (\rho + r_h)^2 d\Omega^2. \quad (15)$$

Now, we impose the horizon-structure-keeping conditions

$$\begin{aligned} \mathcal{L}_\xi g_{\rho\rho} &= -2\partial_\rho \xi^u = 0, \\ \mathcal{L}_\xi g_{u\rho} &= -f(\rho + r_h)\partial_\rho \xi^u - \partial_\rho \xi^\rho - \partial_u \xi^u = 0. \end{aligned} \quad (16)$$

Solving the above Killing equations we obtain

$$\xi^u = F(u, x), \quad \xi^\rho = -\rho\partial_u F(u, x), \quad (17)$$

where  $x$  denotes the coordinates on the unit  $(n-1)$ -sphere. All other components of  $\xi^\mu$  vanish. The condition  $\mathcal{L}_\xi g_{uu} = 0$  is satisfied in the near-horizon limit after a direct calculation. Now, we return to the original coordinates  $(t, \rho)$ , the vector fields take the form

$$\xi^t = T - \frac{\rho}{f(\rho + r_h)}\partial_t T, \quad \xi^\rho = -\rho\partial_t T, \quad (18)$$

where  $T(t, \rho, x) \equiv F(u, x)$ .

For the metric (12), the normal vectors can be chosen as

$$N^\mu = \left(0, \sqrt{f(r)}, 0, 0, \dots\right), \quad M^\mu = \left(\frac{1}{\sqrt{f(r)}}, 0, 0, 0, \dots\right). \quad (19)$$

Substituting eqs.(18) and (19) into eq.(8), and inserting the boundary action (11), the Noether charge can be given as

$$Q[\xi] = \frac{1}{8\pi G} \left(1 - \frac{d-1}{d} 2\pi G \phi^2(r_h)\right) \int_{\partial\Sigma} d^{d-1}x \sqrt{h} \left(\kappa T - \frac{1}{2}\partial_t T\right). \quad (20)$$

Note that once  $T$  is determined, the vector field and Noether charge are determined, and they are both linear in  $T$ . We can expand  $T$  in terms of a set of basis functions

$$T = \sum_m A_m T_m, \quad (21)$$

where  $A_m^* = A_{-m}$  in order that  $T$  is real. Since to every  $T_m$  there is a corresponding  $\xi_m^\mu$ , the basis functions  $T_m$  should be chosen properly in order that the Diff  $S^1$  algebra is satisfied:

$$i\{\xi_m, \xi_n\}^\mu = (m-n)\xi_{m+n}^\mu, \quad (22)$$

where  $\{, \}$  is the Lie bracket. The correct form of  $T_m$  is [23]

$$T_m = \frac{1}{\alpha} \exp[im(\alpha t + g(\rho) + p \cdot x)], \quad (23)$$

where  $\alpha$  is a constant,  $p$  is an integer,  $g(\rho)$  is a regular function on the horizon. The expanded modes of  $Q[\xi]$  corresponding to  $T_m$  will henceforth be denoted as  $Q_m$ . The commutators between these modes can be calculated straightforwardly by substituting eqs. (7), (18), (19) into (9),

$$[Q_m, Q_n] = \frac{1}{8\pi G} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2(r_h) \right) \int_{\partial\Sigma} d^{d-1}x \sqrt{h} \left[ \kappa (T_m \partial_t T_n - T_n \partial_t T_m) - \frac{1}{2} (T_m \partial_t^2 T_n - T_n \partial_t^2 T_m) + \frac{1}{4\kappa} (\partial_t T_m \partial_t^2 T_n - \partial_t T_n \partial_t^2 T_m) \right]. \quad (24)$$

Performing the integration, the final form of the modes  $Q_m$  and the commutators are obtained explicitly,

$$Q_m = \frac{1}{8\pi G} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2(r_h) \right) \frac{\kappa A}{\alpha} \delta_{m,0},$$

$$[Q_m, Q_n] = \frac{1}{8\pi G} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2(r_h) \right) \left[ -\frac{i\kappa A}{\alpha} (m-n) \delta_{m+n,0} - im^3 \frac{\alpha A}{2\kappa} \delta_{m+n,0} \right], \quad (25)$$

where  $A$  is the area of the horizon of the black hole.

Then the central term of the Virasoro algebra is

$$K[\xi_m, \xi_n] = [Q_m, Q_n] + i(m-n)Q_{m+n}$$

$$= -im^3 \frac{\alpha A}{2\kappa} \frac{1 - \frac{d-1}{4d} 8\pi G \phi^2(r_h)}{8\pi G} \delta_{m+n,0} \quad (26)$$

The central charge and zero mode energy  $Q_0$  can be read off easily

$$\frac{C}{12} = \frac{A}{16\pi G} \frac{\alpha}{\kappa} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2(r_h) \right), \quad (27)$$

$$Q_0 = \frac{A}{8\pi G} \frac{\kappa}{\alpha} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2(r_h) \right). \quad (28)$$

Using Cardy's formula, we obtain the entropy of the black hole

$$S = 2\pi \sqrt{\frac{CQ_0}{6}} = \frac{A}{4G} \left( 1 - \frac{d-1}{d} 2\pi G \phi^2(r_h) \right). \quad (29)$$

which is exactly the Wald entropy. This result is different from the previous work [14], which adopted the Brown-Henneaux canonical formalism. In [14], under the black hole no hair assumption, Natsuume, Okamura and Sato obtained the same central charge as in pure gravity theories, thus the black hole entropy obtained from the central charge does not agree with the Wald entropy. However, using the MP approach, we showed that the scalar field does contribute to the central charge of the near horizon Virasoro algebra and the Wald entropy is recovered.

## 4 Conclusion

In this paper, we extended the MP approach for constructing near horizon conformal symmetries for pure Einstein gravity theory to the case of gravity with a conformally coupled scalar field. The work presented in this paper confirms that the entropy of black holes in the presence of conformally coupled scalar field is identical to Wald entropy as apposed to previous claim for a discrepancy. Also, our construction does not depend on the concrete choice of the black hole metric (i.e. the function  $f(r)$  remain unspecified), therefore the MP approach can be applied to a broader class of black holes at once, as apposed to the traditional Hamiltonian approach, which depends sensitively on the concrete choice of the black hole solution as well as the near-horizon behavior of each metric elements. We believe that our work is helpful in further extending the application of the MP approach as well as in further understanding the universality of Wald entropy.

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